

Formulaire de calcul tensoriel

1. Tenseurs

Soit $\mathbf{B} = (\vec{e}_i)_{i=1,3}$ base orthonormée directe cartésienne de $\mathbb{R}^3$.

Soit le vecteur $\vec{u} = u_i \vec{e}_i = \sum_{i=1}^3 u_i \vec{e}_i$ (convention de sommation de l'indice répété), représentant la vitesse d'un point par exemple. Soit p un scalaire (pression par exemple).

Le vecteur $\vec{u}$ est un tenseur d'ordre 1. Le tenseur d'ordre 2 est une matrice.

$\bar{\bar{A}} = [A_{ij}]_{i,j=1,2,3}$. Le produit d'une matrice par un vecteur est un vecteur :

$$\vec{v} = \bar{\bar{A}} \cdot \vec{u} = A_{ij} u_j = \begin{pmatrix} \sum_{j=1}^3 A_{1j} u_j \\ \sum_{j=1}^3 A_{2j} u_j \\ \sum_{j=1}^3 A_{3j} u_j \end{pmatrix}$$

Double contraction :

$$\bar{\bar{A}} : \bar{\bar{B}} = A_{ij} B_{ij} = \sum_{i=1}^3 \sum_{j=1}^3 A_{ij} B_{ij} \text{ est un scalaire.}$$

Produit de deux matrice :

$$\bar{\bar{A}} \cdot \bar{\bar{B}} = (A_{ik} B_{kj})_{i,j=1,2,3} \text{ est une matrice.}$$

$$\vec{u} \otimes \vec{v} = [u_i v_j]_{i,j=1,2,3} \text{ est une matrice}$$

2. Opérateurs

2.1. Coordonnées cartésiennes

$$\mathbf{B} = (\vec{x}, \vec{y}, \vec{z})$$

$$\vec{v} = u\vec{x} + v\vec{y} + w\vec{z}$$

Gradient :

$$\overrightarrow{\text{grad}} p = \frac{\partial p}{\partial x} \vec{x} + \frac{\partial p}{\partial y} \vec{y} + \frac{\partial p}{\partial z} \vec{z} \text{ est un vecteur.}$$

$$\overline{\overline{\text{grad}}} \vec{v} = \begin{bmatrix} \overrightarrow{\text{grad}} u \\ \overrightarrow{\text{grad}} v \\ \overrightarrow{\text{grad}} w \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \text{ est une matrice.}$$

Divergence :

$$\text{div } \vec{v} = \text{tr}(\overline{\overline{\text{grad}}} \vec{v}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \text{ est un scalaire.}$$

$$\overrightarrow{\text{div}}(\vec{A}) = \begin{pmatrix} \text{div } \vec{A}_x \\ \text{div } \vec{A}_y \\ \text{div } \vec{A}_z \end{pmatrix} = \begin{pmatrix} \frac{\partial A_{xx}}{\partial x} + \frac{\partial A_{xy}}{\partial y} + \frac{\partial A_{xz}}{\partial z} \\ \frac{\partial A_{yx}}{\partial x} + \frac{\partial A_{yy}}{\partial y} + \frac{\partial A_{yz}}{\partial z} \\ \frac{\partial A_{zx}}{\partial x} + \frac{\partial A_{zy}}{\partial y} + \frac{\partial A_{zz}}{\partial z} \end{pmatrix} = \left(\frac{\partial A_{ij}}{\partial x_j} \right)_{i=1,3} \text{ est un vecteur.}$$

Laplacien :

$$\Delta p = \text{div}(\overrightarrow{\text{grad}} p) = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \text{ est un scalaire}$$

$$\Delta \vec{v} = \overrightarrow{\text{div}}(\overline{\overline{\text{grad}}} \vec{v}) = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{pmatrix} = \left(\frac{\partial^2 u_i}{\partial x_j \partial x_j} \right)_{i=1,3} \text{ est un vecteur.}$$

Rotationnel :

$$\overrightarrow{rot} \vec{v} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \vec{x} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \vec{y} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \vec{z}$$

Taux de déformation :

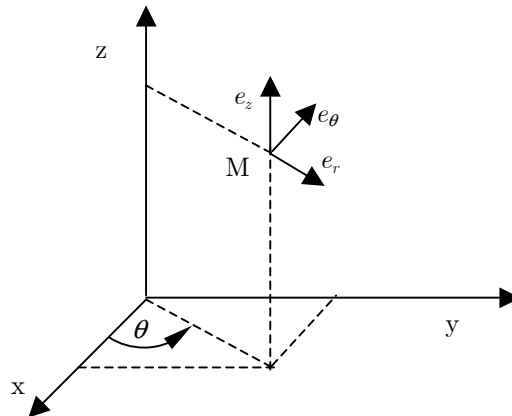
$$\bar{\bar{D}} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ & \frac{\partial v}{\partial y} & \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \\ sym & & \frac{\partial w}{\partial z} \end{bmatrix}$$

Remarque :

L'opérateur gradient augmente l'ordre du tenseur, l'opérateur divergence le diminue et le laplacien le laisse inchangé. Le rotationnel ne s'applique qu'à un vecteur.

2.2. Coordonnées cylindriques

$$B = (\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$$



$$\vec{v} = v_r \vec{e}_r + v_\theta \vec{e}_\theta + v_z \vec{e}_z$$

Gradient :

$$\overrightarrow{grad} p = \frac{\partial p}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial p}{\partial \theta} \vec{e}_\theta + \frac{\partial p}{\partial z} \vec{e}_z$$

$$\overline{\overline{grad}} \vec{v} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r} \frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

Divergence :

$$div \vec{v} = tr(\overline{\overline{grad}} \vec{v}) = \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z}$$

Laplacien :

$$\Delta p = div(\overline{\overline{grad}} p) = \frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2}$$

Rotationnel :

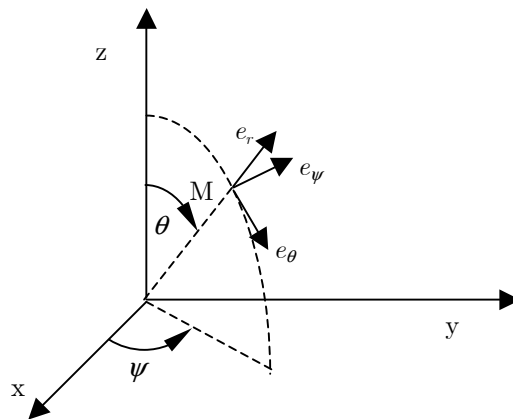
$$\overrightarrow{rot} \vec{v} = \left(\frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \right) \vec{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) \vec{e}_\theta + \frac{1}{r} \left(\frac{\partial(rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta} \right) \vec{e}_z$$

Taux de déformation :

$$\overline{\overline{D}} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{2} \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} \right) & \frac{1}{2} \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \\ & \frac{1}{r} \left(\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) & \frac{1}{2} \left(\frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right) \\ sym & & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

2.3. Coordonnées sphériques

$$B = (\vec{e}_r, \vec{e}_\theta, \vec{e}_\psi)$$



$$\vec{v} = v_r \vec{e}_r + v_\theta \vec{e}_\theta + v_\psi \vec{e}_\psi$$

Gradient :

$$\overrightarrow{grad} p = \frac{\partial p}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial p}{\partial \theta} \vec{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial p}{\partial \psi} \vec{e}_\psi$$

$$\overline{\overline{grad}} \vec{v} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \psi} - \frac{v_\psi}{r} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \psi} - \frac{v_\psi \cot \theta}{r} \\ \frac{\partial v_\psi}{\partial r} & \frac{1}{r} \frac{\partial v_\psi}{\partial \theta} & \frac{1}{r \sin \theta} \frac{\partial v_\psi}{\partial \psi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \end{bmatrix}$$

Divergence :

$$div \vec{v} = tr(\overline{\overline{grad}} \vec{v}) = \frac{1}{r^2} \frac{\partial (r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (v_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial v_\psi}{\partial \psi}$$

Laplacien :

$$\Delta p = div(\overrightarrow{grad} p) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial p}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial p}{\partial \theta} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial^2 p}{\partial \psi^2}$$

Rotationnel :

$$\overrightarrow{rot} \vec{v} = \frac{1}{r^2 \sin \theta} \left(\frac{\partial}{\partial \theta} (r \sin \theta v_\psi) - \frac{\partial (r v_\theta)}{\partial \psi} \right) \vec{e}_r + \frac{1}{r \sin \theta} \left(\frac{\partial v_r}{\partial \psi} - \frac{\partial}{\partial r} (r \sin \theta v_\psi) \right) \vec{e}_\theta + \frac{1}{r} \left(\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta} \right) \vec{e}_\psi$$

Taux de déformation :

$$\overline{\overline{D}} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{2} \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} \right) & \frac{1}{2} \left(\frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \psi} - \frac{v_\psi}{r} + \frac{\partial v_\psi}{\partial r} \right) \\ & \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{1}{2} \left(\frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \psi} + \frac{1}{r} \frac{\partial v_\psi}{\partial \theta} - \frac{v_\psi \cot \theta}{r} \right) \\ sym & & \frac{1}{r \sin \theta} \frac{\partial v_\psi}{\partial \psi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \end{bmatrix}$$

3. Relations usuelles

$$\overrightarrow{\text{grad}}(\Delta p) = \Delta(\overrightarrow{\text{grad}} p)$$

$$\overrightarrow{\text{grad}}(fg) = f\overrightarrow{\text{grad}}(g) + g\overrightarrow{\text{grad}}(f)$$

$$\overrightarrow{\text{grad}}(p\vec{u}) = p\overrightarrow{\text{grad}}(\vec{u}) + \vec{u} \otimes \overrightarrow{\text{grad}} p$$

$$\overrightarrow{\text{grad}}(\vec{u} \cdot \vec{v}) = \left(\overrightarrow{\text{grad}}^T(\vec{u}) \right) \cdot \vec{v} + \left(\overrightarrow{\text{grad}}^T(\vec{v}) \right) \cdot \vec{u}$$

$$\left(\overrightarrow{\text{grad}}(\vec{u}) \right) \cdot \vec{u} = \overrightarrow{\text{grad}}\left(\frac{u^2}{2}\right) + (\overrightarrow{\text{rot}} \vec{u}) \wedge \vec{u}$$

$$\overrightarrow{\text{grad}}(\vec{u}) = \overline{\overline{D}} + \overline{\overline{\Omega}}, \quad \overline{\overline{D}} = \frac{1}{2} \left(\overrightarrow{\text{grad}}(\vec{u}) + \overrightarrow{\text{grad}}^T(\vec{u}) \right), \quad \overline{\overline{\Omega}} = \frac{1}{2} \left(\overrightarrow{\text{grad}}(\vec{u}) - \overrightarrow{\text{grad}}^T(\vec{u}) \right)$$

$$\text{div } \vec{u} = \text{tr}(\overrightarrow{\text{grad}} \vec{u})$$

$$\text{div}(\Delta \vec{u}) = \Delta(\text{div } \vec{u})$$

$$\text{div}(\overrightarrow{\text{rot}} \vec{u}) = 0$$

$$\text{div}(p\vec{u}) = p\text{div } \vec{u} + \vec{u} \cdot \overrightarrow{\text{grad}} p$$

$$\overrightarrow{\text{div}}(p\overline{\overline{A}}) = p\overrightarrow{\text{div}}(\overline{\overline{A}}) + \overline{\overline{A}} \cdot \overrightarrow{\text{grad}} p$$

$$\overrightarrow{\text{div}}(p\overline{\overline{I}}) = \overrightarrow{\text{grad}} p$$

$$\text{div}(\vec{u} \wedge \vec{v}) = \vec{v} \cdot \overrightarrow{\text{rot}} \vec{u} - \vec{u} \cdot \overrightarrow{\text{rot}} \vec{v}$$

$$\overrightarrow{\text{div}}(\vec{u} \otimes \vec{v}) = \vec{u} \text{div } \vec{v} + \left(\overrightarrow{\text{grad}} \vec{u} \right) \cdot \vec{v}$$

$$\overrightarrow{\text{div}}(\overline{\overline{D}}) = \Delta \vec{u} + \frac{1}{2} \overrightarrow{\text{rot}}(\overrightarrow{\text{rot}} \vec{u}) = \frac{1}{2} \overrightarrow{\text{grad}}(\text{div } \vec{u}) + \frac{1}{2} \Delta \vec{u}$$

$$\overrightarrow{\text{div}}(\overline{\overline{\Omega}}) = -\frac{1}{2} \overrightarrow{\text{rot}}(\overrightarrow{\text{rot}} \vec{u})$$

$$\Delta p = \text{div}(\overrightarrow{\text{grad}} p)$$

$$\Delta \vec{u} = \overrightarrow{\text{div}}(\overrightarrow{\text{grad}} \vec{u}) = \overrightarrow{\text{grad}}(\text{div } \vec{u}) - \overrightarrow{\text{rot}}(\overrightarrow{\text{rot}} \vec{u})$$

$$\Delta(fg) = f\Delta g + g\Delta f + 2\overrightarrow{\text{grad}} f \cdot \overrightarrow{\text{grad}} g$$

$$\Delta(p\vec{u}) = p\Delta \vec{u} + \vec{u}\Delta p + 2\left(\overrightarrow{\text{grad}} \vec{u} \right) \cdot \overrightarrow{\text{grad}} p$$

$$\overrightarrow{\text{rot}}(\overrightarrow{\text{grad}} p) = \vec{0}$$

$$\overrightarrow{\text{rot}}(\Delta \vec{u}) = \Delta(\overrightarrow{\text{rot}} \vec{u})$$

$$\overrightarrow{\text{rot}}(p\vec{u}) = p\overrightarrow{\text{rot}} \vec{u} + \left(\overrightarrow{\text{grad}} p \right) \wedge \vec{u}$$

$$\overrightarrow{\text{rot}}(\vec{u} \wedge \vec{v}) = \vec{u} \text{div } \vec{v} - \vec{v} \text{div } \vec{u} + \left(\overrightarrow{\text{grad}} \vec{u} \right) \cdot \vec{v} - \left(\overrightarrow{\text{grad}} \vec{v} \right) \cdot \vec{u}$$

4. Equations de Navier-Stokes en incompressible

Continuité :

$$\operatorname{div} \vec{v} = 0$$

Navier-Stokes (Quantité de mouvement) :

$$\frac{\partial \vec{v}}{\partial t} + \left(\overline{\operatorname{grad}} \vec{v} \right) \vec{v} = \frac{1}{\rho} \vec{f} - \frac{1}{\rho} \overline{\operatorname{grad}} p + \nu \Delta \vec{v} \quad , \quad \nu = \frac{\mu}{\rho}$$

4.1. Coordonnées cartésiennes

Continuité :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

N.S. :

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= \frac{1}{\rho} f_x - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= \frac{1}{\rho} f_y - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right] \\ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= \frac{1}{\rho} f_z - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right] \end{aligned}$$

4.2. Coordonnées cylindriques

Continuité :

$$\frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r} = 0$$

N.S. :

$$\begin{aligned}
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} &= \frac{1}{\rho} f_r - \frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left[\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} - \frac{v_r}{r^2} \right] \\
\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} - \frac{v_r v_\theta}{r} &= \frac{1}{\rho} f_\theta - \frac{1}{\rho} \frac{1}{r} \frac{\partial p}{\partial \theta} \\
&+ \nu \left[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2} \right] \\
\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} &= \frac{1}{\rho} f_z - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left[\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} \right]
\end{aligned}$$

4.3. Coordonnées sphériques

Continuité :

$$\frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial v_\psi}{\partial \psi} + \frac{2v_r}{r} + \frac{v_\theta \cot \theta}{r} = 0$$

N.S. :

$$\begin{aligned}
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\psi}{r \sin \theta} \frac{\partial v_r}{\partial \psi} - \frac{v_\theta^2 + v_\psi^2}{r} &= \frac{1}{\rho} f_r - \frac{1}{\rho} \frac{\partial p}{\partial r} \\
+ \nu \left[\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \psi^2} + \frac{2}{r} \frac{\partial v_r}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial v_\psi}{\partial \psi} - \frac{2v_r}{r^2} - \frac{2v_\theta \cot \theta}{r^2} \right] \\
\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\psi}{r \sin \theta} \frac{\partial v_\theta}{\partial \psi} + \frac{v_r v_\theta}{r} - \frac{v_\psi^2 \cot \theta}{r} &= \frac{1}{\rho} f_\theta - \frac{1}{\rho} \frac{1}{r} \frac{\partial p}{\partial \theta} \\
+ \nu \left[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \psi^2} + \frac{2}{r} \frac{\partial v_\theta}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial v_\theta}{\partial \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial v_\psi}{\partial \psi} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2 \sin^2 \theta} \right] \\
\frac{\partial v_\psi}{\partial t} + v_r \frac{\partial v_\psi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\psi}{\partial \theta} + \frac{v_\psi}{r \sin \theta} \frac{\partial v_\psi}{\partial \psi} + \frac{v_r v_\psi}{r} + \frac{v_\theta w \cot \theta}{r} &= \frac{1}{\rho} f_\psi - \frac{1}{\rho r \sin \theta} \frac{\partial p}{\partial \psi} \\
\nu \left[\frac{\partial^2 v_\psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_\psi}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\psi}{\partial \psi^2} + \frac{2}{r} \frac{\partial v_\psi}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial v_\psi}{\partial \theta} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial v_\theta}{\partial \psi} + \frac{2}{r^2 \sin^2 \theta} \frac{\partial v_r}{\partial \psi} - \frac{v_\psi}{r^2 \sin^2 \theta} \right]
\end{aligned}$$